Reply to "Comment on 'Nonlinear resonance and chaos in the relativistic phase space for driven nonlinear systems' "

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As a Reply to the Comment by Luchinsky *et al.* [preceding paper, Phys. Rev. E **53**, 4240 (1996)] on the paper by Kim and Lee [Phys. Rev. E **52**, 473 (1995)], we point out that the formation of zero-dispersion nonlinear resonance can always be observed in a relativistic hard oscillator driven by a periodic force. We also report results of a computation of resonance energies for a driven relativistic Duffing oscillator using the formula suggested by Luchinsky *et al.*

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Luchinsky *et al.* [1] noted that what they referred to as zero-dispersion nonlinear phenomena can be observed in some of driven relativistic nonlinear oscillators studied by Kim and Lee [2]. We wish to point out that zero-dispersion nonlinear resonance is indeed expected to always occur for the case of a relativistic "hard" oscillator [$V(q)\sim|q|^n$ with $n>2$] driven by a periodic external force. At nonrelativistic energies, the frequency $\Omega(E)$ of a hard oscillator is an increasing function of the oscillator energy E. At ultrarelativistic energies, however, the frequency $\Omega(E)$ becomes a decreasing function of energy E. The dispersion $\frac{d\Omega(E)}{dE}$ should thus vanish at some relativistic energy E. As Luchinsky *et al.* pointed out, this leads to an appearance of zero-dispersion nonlinear resonance.

Luchinsky *et al.* also suggested that Eq. (2) of Ref. [1]

$$n=\frac{\omega}{\Omega}\pm nF_0\frac{dq_n}{dE}, \tag{1}$$

where ω and F_0 are, respectively, the frequency and amplitude of the driving force and q_n are coefficients in the Fourier expansion of q [$q=2\Sigma_{n=0}^{\infty}q_n(E)\cos n\psi$, ψ is the angle variable], yields a more accurate determination of resonance energies than Eq. (3) of Ref. [2]

$$n=\frac{\omega}{\Omega}. \tag{2}$$

While no significant improvement is expected for the case of a relatively weak force investigated by Kim and Lee, it should still be of interest to see if the discrepancy between theoretical and numerical data of Kim and Lee can be resolved by Eq. (1). We thus performed computation to determine resonance energies for the model system of the Duffing oscillator [$V(q)=-\frac{1}{4}q^2+\frac{1}{8}q^4$] using Eq. (1). In Table I we show results of our computation where parameters are chosen to be m (mass of the oscillator)=1, ω (driving frequency)=$\pi/2$, and c (speed of light)=5.5. Numerical values as well as theoretical values obtained using Eq. (2) were already reported in Ref. [2] (see Table II of Ref. [2]). Theoretical values obtained using Eq. (1), represent results of our new computation. It should be noted that Eq. (1), with the $\pm$ sign on the right-hand side, yields two values of resonance energy, one at the elliptic fixed point and one at the hyperbolic fixed point. Resonance energies shown in Table I represent energies at the elliptic fixed points. In our computation using Eq. (1), we therefore chose the sign that yields energy at the elliptic fixed point of each resonance. We indicate our choice of sign in parentheses in the last column of Table I. It can be seen clearly from Table I that Eq. (1) yields a more accurate determination of the resonance energies than Eq. (2) does.

TABLE I. Resonance energies of the driven relativistic Duffing oscillator computed numerically and calculated theoretically using, respectively, Eq. (2) and Eq. (1). Parameters are chosen to be m (mass of the oscillator)=1, ω (driving frequency)=$\frac{\pi}{2}$, and c (speed of light)=5.5. n refers to the period-n resonance and F_0 is the amplitude of the driving force. The signs in the parentheses in the last column indicate our choice of sign on Eq. (1).

		Resonance energy		
n	F_0	Numerical	Theoretical Eq. (2)	Theoretical Eq. (1)
5	0.01	0.004	0.007	0.004 (+)
3	0.01	0.100	0.103	0.101 (+)
1	0.1	15.64	15.13	15.63 (+)
1	0.1	54.24	53.91	54.23 (−)

[1] D. G. Luchinsky, P. V. E. McClintock, S. M. Soskin, and N. D. Stein, preceding paper, Phys. Rev. E **53**, 4240 (1996).
[2] J. H. Kim and H. W. Lee, Phys. Rev. E **52**, 473 (1995).

